NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

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PRACTICAL METHODS OF CALCULATION INVOLVED IN THE EXPERIMENTAL STUDY OF AN AUTOPILOT AND THE AUTOPILOT-AIRCRAFT COMBINATION

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SUMMARY

Practical methods are presented for making the various calculations required for the analysis of an autopilot and an autopilot-aircraft combination from frequency-response data. Equations are derived for determining the servo-system error voltage for both displacement input signal and displacement plus rate of displacement input signals, the autopilot frequency response for addition of rate of displacement input signal, the servo-system frequency response for a change of gain, and the relation between open-loop and closed-loop frequency responses for the servo system and for the autopilot-aircraft combination. Where possible, comparisons are made between experimental data and calculated responses using the equations developed.

INTRODUCTION

The concept of predicting the dynamic stability of an autopilot-controlled airplane from the individual frequency responses of the autopilot and aircraft is well known. The basic theory may be readily obtained from texts on servomechanisms such as reference 1 and has been applied in several NACA reports, of which reference 2 gives a comprehensive survey of the techniques developed and a bibliography of the field. However, in the course of conducting experimental work to evaluate and analyze the performance of a particular autopilot-aircraft combination it was found necessary to derive from the basic theory several analytical methods and formulas for handling experimental data. The combination studied is typical of autopilot-aircraft systems, being of the position-control type which is characterized by feedback of angular displacement and rate of angular displacement. Hence, the methods and formulas should prove useful to others investigating similar systems. A block diagram showing the components of such a system is given in figure 1.

One limitation to predicting stability by the usual methods is that the theory applies only if the entire system is linear in operation. In practice, the system is linear only over a limited range of operation and it becomes important to know the extent of this range in order that data may be obtained for the system in linear operation. A common source of

nonlinearity occurs in the servo system and is referred to as saturation. When a certain level of servo-system error voltage (voltage input to the servo amplifier) is exceeded, one or more components exhibit saturation and the output-input relation is no longer linear. Although the error voltage can be measured directly when obtaining the frequency response of a closed-loop system, frequently it is easier to determine its limiting value by making a few sample calculations using the input and output data. Formulas are derived for the calculation of error voltage for either displacement signal or displacement plus rate of displacement signal. Thus, input signals to the autopilot may be chosen such that the servo-system error voltage does not exceed the value which would cause partial saturation, and the entire system will operate in the linear range.

When the autopilot is combined with the aircraft, an additional signal, the command input signal to the combination, affects the error voltage to the servo amplifier. The error voltage for this case is treated in the discussion of the complete system.

The most accurate way of determining the frequency response of the autopilot where both displacement and rate gyros are used is to mechanically oscillate the gyros. Not only does this require an oscillating table drive but a great many tests have to be made to adequately cover the range of possible values of displacement and rate. Hence it may prove expedient to calculate the rate response for any desired amount of rate signal from the gyro characteristics and the measured servo-system response for displacement signal only. The response for displacement signal only can be readily determined using a sine-wave generator to simulate the electrical signal from the displacement gyro. From relatively simple measurements of the rate gyro its steady-state characteristics may be obtained. The natural frequency and damping ratios are usually such that the variations of phase angle and amplitude ratio are negligible over the frequency range of interest for the autopilot-airplane combination. The method for computing from the above data the autopilot frequency response for various amounts of displacement and rate of displacement is given.

In a stability analysis of the entire system, conditions for combining autopilot and aircraft responses might necessitate obtaining a servo response at some particular value of servo-loop gain for which tests were not made. This may be the case when the stability of the autopilot-aircraft combination is controlled in part by the servo follow-up potentiometer which at the same time alters the servo-system gain. It then becomes convenient to be able to calculate the servo response at this new value of gain using the data obtained with some other value of gain. Equations are derived for performing this operation directly without the usual necessity of converting the closed-loop servo response to open-loop, changing gain, and converting back to closed-loop.

Finally, in working with any closed-loop system, it is usually necessary to know the relations between open-loop and closed-loop conditions. In the case of the servo system for which the closed-loop response is determined, ordinarily it is desirable to know approximately the form or order of the equation associated with the system. This information can be seen from a plot of the open-loop response. As far as the autopilot-aircraft combination is concerned, it is often necessary to determine the closed-loop response from measurements of the component responses. Conversely, if the closed-loop response is measured in flight, it is generally desired to convert to the open-loop response to check on the relative stability. The relations for carrying out these analyses are given.

DEFINITIONS AND SYMBOLS

Frequency response: A frequency-dependent vector response of the output of a system to a sinusoidally varying input function, expressed quantitatively by a plot of amplitude ratio and phase angle versus frequency

Amplitude ratio: The ratio of the output amplitude to the input amplitude. For a closed-loop system this is ordinarily converted to dimensionless form by dividing by the amplitude ratio at zero frequency

Phase angle: The angle between an output vector and input vector. When the output leads the input, the angle is positive

Closed-loop response: The frequency response of a closed-loop system, that is, one which possesses feedback and is sensitive to the difference between output and input

Open-loop response: The frequency response of an open-loop system

Servo system: That part of the autopilot composed of the amplifier and servo actuator or motor and its own feedback loop

Autopilot: The aircraft stabilizing device composed of the servo system, the error-measuring component, and other feedback elements

Voltages, angular displacements, and transfer functions: Vector quantities having amplitudes and phase angles, unless otherwise noted

A open-loop transfer function of servo system

A_a transfer function of servo amplifier

A_L open-loop transfer function of autopilot-aircraft combination

$$\left(\frac{v_g+v_r}{v_i}\right) \text{ or } \left(\frac{\theta+\theta_r}{\theta_i}\right)$$

A_m transfer function of servo motor or actuator

A_r transfer function of rate gyro

e 2.718...

f frequency, cycles per second

j $\sqrt{-1}$

k_f follow-up pickoff constant, volts per degree

k_g displacement gyro constant, volts per degree

k_p static control gearing, ratio of control-surface deflection to angular-displacement input to autopilot, degrees per degree

k_r rate-gyro constant, volts per cycle per second per degree oscillation

N ratio of two open-loop transfer functions with different values of gain

P_a gain of amplifier attenuator, percent

P_f gain of follow-up attenuator, percent

P_r gain of rate-gyro attenuator, percent

R amplitude ratio of closed-loop servo response, dimensionless

R_{fr} amplitude ratio of autopilot response when rate of displacement input signal is included, dimensionless

(t) function of time

v_e error signal of servo system, input to amplifier attenuator, volts

v_{ec} error signal of servo system for the autopilot-aircraft combination, volts

v_{er} error signal of servo system when rate of displacement input signal is added, volts

v_E	error signal of autopilot-aircraft combination $\left(v_I - v_g\right)$, volts
v_f	feedback voltage of servo system, volts
v_{fr}	feedback voltage of servo system when rate of displacement input signal is added, volts
v_g	displacement gyro output, volts
v_i	input signal to servo system, volts
v_I	input signal to autopilot-aircraft combination, volts
v_r	rate gyro output, modified by rate attenuator, volts
δ	control surface deflection, degrees
ϵ_e	phase angle of v_e relative to v_i, degrees
ϵ_{er}	phase angle of v_{er} relative to θ, degrees
ϵ_f	phase angle of v_f relative to v_i, degrees
ϵ_{fe}	phase angle of v_f relative to v_e, degrees
ϵ_{fr}	phase angle of v_{fr} relative to θ, degrees
ϵ_L	phase angle of (v_g+v_r) relative to v_i (and $\theta+\theta_r$ relative to θ_i) when the autopilot-aircraft loop is opened, degrees
ϵ_r	phase angle of v_r relative to θ, degrees
θ	angular displacement, attitude of aircraft, degrees
θ_E	error angle, degrees
θ_i	hypothetical input angle to servo system, degrees
θ_I	input angle to autopilot-aircraft combination, degrees
θ_r	hypothetical rate feedback angle, degrees
ω	angular frequency, radians per second

See diagram on page 18.

Subscript

max	maximum amplitude of a variable denoting a scalar quantity

ANALYSIS

Servo-System Error Voltage

Displacement signal only.- As mentioned previously, for purposes of analysis it is necessary to limit the error voltage to a value which will allow the servo system to operate within the linear range. The error voltage may be calculated using the measured closed-loop frequency response as follows:

With reference to figure 1, the error voltage for the servo system alone is the difference between the input and follow-up (feedback) voltages, or

$$v_e = v_i - v_f \tag{1}$$

where v_e and v_f are vector quantities with phase angles measured relative to the input vector v_i.

The closed-loop response of the servo system is δ/v_i but in dimensionless form is equivalent to v_f/v_i since $v_f = P_f k_f \delta$. The nondimensional form is preferred since it is easier to obtain experimentally. Furthermore, from the standpoint of the calculations involved in an analysis it is more convenient to separate the actual response into a frequency-variant characteristic (the nondimensional form) and a constant quantitatively relating the output to input under static conditions. The amplitude ratio and phase angle of the closed-loop response may be represented by R and ϵ_f, respectively, so that

$$\frac{v_f}{v_i} = Re^{j\epsilon_f} \tag{2}$$

or

$$v_f = v_i Re^{j\epsilon_f} \tag{3}$$

then

$$v_e = v_i - v_i Re^{j\epsilon_f}$$

$$= v_i (1-R\cos\epsilon_f - j R\sin\epsilon_f) \tag{4}$$

$$= v_i \sqrt{1 + R^2 - 2R\cos\epsilon_f} \quad e^{j\epsilon_e} \tag{5}$$

where

$$\epsilon_e = -\tan^{-1}\frac{R\sin\epsilon_f}{1-R\cos\epsilon_f} \tag{6}$$

Inspection of equation (5) shows that for a given input signal the magnitude of the error voltage increases as the magnitudes of the closed-loop response and phase angle increase. Thus, the error voltage will be small at very low frequencies and increase to a peak near the resonant frequency. Ultimately, at high frequencies where the response magnitude diminishes to zero, the error voltage approaches the input voltage in magnitude.

Displacement plus rate of displacement input signal.- The response to displacement plus rate of displacement input signals may be determined by sinusoidally oscillating the displacement and rate gyros, considering θ as the input. To compute the error voltage for this condition it is first necessary to develop an expression for the rate signal. If the rate gyro is assumed to be rocking with a motion

$$\theta(t) = \theta_{max} \sin \omega t$$

the corresponding rate equation is

$$\frac{d\theta(t)}{dt} = \omega\theta_{max} \cos \omega t \tag{7}$$

For the frequency range of interest, gyro resonance effects are nearly always negligible so that the output voltage is related to the rate of angular motion of equation (7) by a constant k_r:

$$v_r(t) = k_r f \; \theta_{max} \cos \omega t$$

Usually this output is modified by an attenuation factor P_r that governs the amount of rate signal, that is,

$$v_r(t) = P_r k_r f \theta_{max} \cos \omega t \tag{8}$$

Since $\theta_{max} \cos \omega t$ represents the vector θ shifted in phase $\pi/2$ radians, equation (8) may be written in vector form as

$$v_r = P_r k_r f \theta e^{j\frac{\pi}{2}}$$

In the practical case the rate-gyro phase angle may not be 90° as indicated above. Hence, the general symbol ϵ_r will be used for phase angle. Then

$$v_r = P_r k_r f \theta e^{j\epsilon_r} \tag{9}$$

$$= P_r A_r \theta \tag{10}$$

in which $A_r = k_r f e^{j\epsilon_r}$ is the transfer function of the rate gyro.

The autopilot response is defined as the control-surface motion per unit angular input to the gyros. For the case of displacement plus rate of displacement input signals the autopilot response is $(\delta/\theta)_r$. The addition of the subscript r here and in the following equations indicates that the quantities are measured with rate signal present. The quantity v_g may be considered the equivalent electrical input to the autopilot since it is directly related by a constant k_g to the actual input θ and is in phase with θ, dynamic effects of a displacement gyro being negligible in the range of frequencies considered. The term v_{fr} may be used to represent the equivalent electrical output of the autopilot when both displacement and rate gyros are sinusoidally oscillated. Hence, the autopilot response may be designated as v_{fr}/v_g which is the nondimensional form usually obtained from measurements. Resolving into amplitude and phase components, this response may be written

$$\frac{v_{fr}}{v_g} = R_{fr}\, e^{j\epsilon_{fr}} \tag{11}$$

The basic equation for the error voltage is

$$v_{er} = v_g + v_r - v_{fr}$$

The substitution for v_r from equation (9) and v_{fr} from equation (11) gives

$$v_{er} = v_g + P_r k_r f\theta e^{j\epsilon_r} - v_g R_{fr}\, e^{j\epsilon_{fr}}$$

By substitution of v_g/k_g for θ

$$\begin{aligned} v_{er} &= v_g + \frac{P_r k_r f v_g}{k_g} e^{j\epsilon_r} - v_g R_{fr}\, e^{j\epsilon_{fr}} \\ &= v_g \left[\left(1 - R_{fr} \cos \epsilon_{fr} + \frac{P_r k_r f}{k_g} \cos \epsilon_r\right) - \right. \\ &\qquad \left. j\left(R_{fr} \sin \epsilon_{fr} - \frac{P_r k_r f}{k_g} \sin \epsilon_r\right)\right] \end{aligned} \tag{12}$$

The magnitude of the error voltage could be found from this equation but if, as is usually the case, the rate-gyro phase angle is close to 90° the error voltage simplifies to

$$\begin{aligned} v_{er} &= v_g \left[(1 - R_{fr} \cos \epsilon_{fr}) - j\left(R_{fr} \sin \epsilon_{fr} - \frac{P_r k_r f}{k_g}\right)\right] \\ &= v_g (E + jF) \end{aligned} \tag{13}$$

from which the magnitude, of the only interest here, is

$$|v_{er}| = |v_g|\sqrt{E^2 + F^2} \tag{14}$$

The expression for the error voltage existing in the autopilot-aircraft closed loop is derived in a later section covering the autopilot-aircraft combination.

Autopilot Response With Displacement Plus Rate of Displacement Input Signal

From the experimental closed-loop response for displacement signal only, it is possible to calculate the autopilot response for any amount of rate signal within the linear operating range. The resultant response may be used in autopilot-aircraft loop calculations as shown in later sections. The autopilot response as defined previously is the control-surface motion per unit angular input to the gyros and may be represented by $(\delta/\theta)_r$. Here, again, the subscript r is used to denote the condition when rate signal is present. Then, referring to figure 1,

$$\left(\frac{\delta}{\theta}\right)_r = \left(\frac{\delta}{v_i}\right)\left(\frac{v_i}{\theta}\right)_r$$

$$= \left(\frac{\delta}{v_i}\right)\left(k_g + P_r A_r\right)$$

In order to obtain the responses in nondimensional terms, substitutions are made for δ and θ from the relations $v_f = k_f P_f \delta$ and $v_g = k_g \theta$. Hence,

$$\left(\frac{v_f}{k_f P_f}\frac{k_g}{v_g}\right)_r = \frac{v_f}{k_f P_f v_i}\left(k_g + P_r A_r\right)$$

Simplifying,

$$\left(\frac{v_f}{v_g}\right)_r = \frac{v_f}{v_i}\left(\frac{k_g + P_r A_r}{k_g}\right)$$

The term $(v_f/v_g)_r$, representing the desired nondimensional response of the autopilot, may be written as v_{fr}/v_g. Therefore

$$\frac{v_{fr}}{v_g} = \frac{v_f}{v_i}\left(1 + \frac{P_r A_r}{k_g}\right) \tag{15}$$

Equation (15) gives the autopilot response for displacement plus rate of displacement in terms of the servo-system response and the relative amount of rate to displacement signal. This equation can be expanded

to give the amplitude and phase responses:

$$\frac{v_{fr}}{v_g} = Re^{j\epsilon_f}\left(1 + \left|\frac{P_rA_r}{k_g}\right| e^{j\epsilon_r}\right)$$

$$= Re^{j\epsilon_f}\left[\left(1 + \left|\frac{P_rA_r}{k_g}\right| \cos \epsilon_r\right) + j\left|\frac{P_rA_r}{k_g}\right| \sin \epsilon_r\right]$$

$$= R_{fr}e^{j\epsilon_{fr}}$$

from which

$$R'_{fr} = R\sqrt{1 + \left|\frac{P_rA_r}{k_g}\right|^2 + 2\left|\frac{P_rA_r}{k_g}\right| \cos \epsilon_r} \tag{16}$$

$$\epsilon_{fr} = \epsilon_f + \tan^{-1}\frac{\left|\frac{P_rA_r}{k_g}\right| \sin \epsilon_r}{1 + \left|\frac{P_rA_r}{k_g}\right| \cos \epsilon_r} \tag{17}$$

If the rate-gyro phase angle is constant at 90°, equations (16) and (17) reduce to

$$R_{fr} = R\sqrt{1 + \left|\frac{P_rA_r}{k_g}\right|^2} \tag{18}$$

$$\epsilon_{fr} = \epsilon_f + \tan^{-1}\left|\frac{P_rA_r}{k_g}\right| \tag{19}$$

A comparison of calculated with experimental values of autopilot frequency response for combined displacement plus rate of displacement input is given in figure 2. The amplitude ratio and phase-angle curves for zero-rate signal were obtained experimentally by running a frequency response on a typical autopilot servo system. By substitution in equations (16) and (17) of the data obtained from the zero rate signal curves, the response curves for two values of P_r were obtained. (The values of $P_r = 8$ percent and 20 percent gave values for P_rA_r/k_g of 0.83f and 2.07f, respectively, up to a frequency of 1.2 cycles per second. At higher frequencies the amplitude of A_r departed from its linear relationship with frequency and the actual measured values were used in the calculations.) The experimental points shown for the rate

signals were obtained by oscillating displacement and rate gyros sinusoidally and feeding their electrical outputs simultaneously to a servo system.

It is seen that agreement between calculated and experimental values is very good up to a frequency of about 1.5 cycles per second. The reason for the dropoff of experimental values beyond this frequency is found in the saturation of the servo amplifier with relatively large error voltages. The error voltage for each condition of rate was calculated from equation (14) and is shown at the bottom of figure 2. The nonlinearity level is indicated to show the point at which saturation of the amplifier begins. It is seen that the amplifier begins to saturate in each case at about the frequency at which the experimental response-amplitude-ratio values start to fall off.

Closed-Loop Servo-System Response for Any Value of Gain

Closed-loop frequency-response tests of servo systems are generally made at several values of gain, but it is obviously impractical to conduct measurements at all possible values. It is helpful to have a method for calculating the response at any value of gain from the response at some particular gain setting. The usual, and laborious, method is to convert the closed-loop response to the equivalent open-loop response, change the gain to the desired value, and then calculate the new closed-loop response. The method derived here gives the desired response directly in terms of the original response and the ratio of gain values.

The gain of an open-loop system is defined as the frequency invariant portion of the open-loop transfer function. This transfer function is the product of the individual component transfer functions and for the servo system in figure 1 is $P_aA_aA_mk_fP_f$. The symbols A_a and A_m represent the complex transfer functions of the amplifier and motor or actuator, respectively. The feedback-pickoff constant is k_f, while P_a and P_f represent the values of gain associated with the amplifier input and follow-up attenuators. It is by means of either of these two attentuators that the gain of the system is commonly varied in operation, and their effect on the open-loop response is independent of frequency.

For a given condition of gain denoted by the subscript 1, the closed-loop response from elementary servo theory is given by

$$\frac{v_{f_1}}{v_i} = \frac{A_1}{1+A_1} \tag{20}$$

where A_1 is the open-loop transfer function for condition 1 of either P_a or P_f. Similarly, for the new desired condition 2,

$$\frac{v_{f_2}}{v_i} = \frac{A_2}{1+A_2} \tag{21}$$

It is convenient to define N as the ratio of gains, that is,

$$N = \frac{A_2}{A_1} \tag{22}$$

which is simply the ratio of P_{a_2} to P_{a_1} or P_{f_2} to P_{f_1} if all other components are held unchanged in value. Substituting A_2 from equation (22) into (21), thus eliminating A_2,

$$\frac{v_{f_2}}{v_i} = \frac{NA_1}{1+NA_1} \tag{23}$$

From equation (20)

$$\frac{v_{f_1}}{v_i} + A_1 \left(\frac{v_{f_1}}{v_i}\right) = A_1$$

and

$$A_1 = \frac{v_{f_1}/v_i}{1-(v_{f_1}/v_i)}$$

which, when substituted into equation (23), gives

$$\frac{v_{f_2}}{v_i} = \frac{N\left(\frac{v_{f_1}}{v_i}\right)}{1 - \frac{v_{f_1}}{v_i} + N\left(\frac{v_{f_1}}{v_i}\right)} \tag{24}$$

The transfer function v_{f_1}/v_i is the complex vector $R_1e^{j\epsilon_{f_1}}$. Substituting this expression for v_{f_1}/v_i in equation (24),

$$\frac{v_{f_2}}{v_i} = \frac{N\,R_1e^{j\epsilon_{f_1}}}{1 - R_1e^{j\epsilon_{f_1}} + N\,R_1e^{j\epsilon_{f_1}}}$$

$$= \frac{N\,R_1e^{j\epsilon_{f_1}}}{\left[1 + R_1\,(N-1)\,\cos\,\epsilon_{f_1}\right] + j\left[R_1\,(N-1)\,\sin\,\epsilon_{f_1}\right]} \tag{25}$$

$$\frac{v_{f_2}}{v_i} = \frac{N\,R_1 e^{j\epsilon_{f_2}}}{\sqrt{X^2+Y^2}} \tag{26}$$

where

$$\epsilon_{f_2} = \epsilon_{f_1} - \tan^{-1}\frac{Y}{X} \tag{27}$$

and X and Y are the real and imaginary parts, respectively, of the denominator of equation (25).

A comparison of calculated with experimental values of frequency response for a change of gain is given in figure 3. By substitution in equations (26) and (27) of the values for amplitude ratio and phase angle from the zero-rate signal curves of figure 2, a new response was obtained for a value of gain differing from the original value by a factor of 2.17. The curves for this response are shown along with points determined experimentally by operating the servo system at the increased value of gain. Agreement is seen to be quite good.

Open-Loop and Closed-Loop Relations

General concepts of open-loop and closed-loop responses and their interrelation have been treated in servomechanism literature. It is the purpose in this section to apply these relations to a typical autopilot-aircraft system as diagramed in figure 1. It can be seen from this figure that two closed-loop systems are in evidence.

The first or inner loop is the servo system alone with v_i as the input, δ as the output, and v_f as the feedback path. Once its closed-loop response has been determined, the servo system may be represented by a single "black box," provided it is stable, and it becomes one of the components in the outer loop. The outer or autopilot-aircraft loop, then, consists of the servo system and aircraft in series in the forward part of the loop and the displacement and rate gyros in the feedback path.

It should be noted that, although the servo system is a relatively simple loop, the combination is not, since it contains a dynamic element in the feedback path. For the servo system the responses δ/v_i and v_f/v_i are dynamically the same, differing only by a constant $P_f k_f$. For the autopilot-aircraft combination, however, the closed-loop responses $(v_g+v_r)/v_I$ and θ/v_I differ dynamically due to the feedback term A_r which is frequency dependent. Hence, in this case, the open-loop response can be obtained from the flight closed-loop response θ/v_I only if the rate feedback transfer function $P_r A_r$ is also known.

In analyzing an existing closed-loop system it is generally not feasible to measure the open-loop response directly. This is due to the very large output magnitudes obtained at low frequencies and because of the inherent drift in the components. Therefore the open-loop response, when required, is calculated from the corresponding measured closed-loop response. Conversely, when a system is synthesized from the responses of the several components, it is the open-loop response which is obtained directly. If the corresponding closed-loop response is desired, it is then necessary to calculate it from the open-loop response.

The open-loop and closed-loop relations for the servo system are given mainly for completeness since they are already thoroughly treated in the literature. The relations for the autopilot-aircraft combination in terms of quantities experimentally determined are not well known and are fully developed in the following paragraphs. It is, of course, necessary that the calculations be restricted to the linear operating range of the various components which is limited by the value of servo-system error voltage at which saturation of one of the components begins to occur. The expression for the error voltage for the autopilot-aircraft combination is therefore derived.

Servo system.- With reference to figure 1, the open-loop response of the servo system alone is v_f/v_e and is designated by the symbol A. The nondimensional closed-loop response is v_f/v_i and is given in terms of the open-loop response by the relation

$$\frac{v_f}{v_i} = \frac{A}{1+A} \tag{28}$$

The open-loop response in terms of the measured closed-loop response is then given as

$$\frac{v_f}{v_e} = A = \frac{v_f/v_i}{1 - (v_f/v_i)} \tag{29}$$

$$= \frac{Re^{j\epsilon_f}}{1 - Re^{j\epsilon_f}}$$

$$= \left[\frac{R}{\sqrt{(1-R\cos\epsilon_f)^2 + (R\sin\epsilon_f)^2}}\right] e^{j\epsilon_{f_e}}$$

$$= \left(\frac{R}{\sqrt{1+R^2-2R\cos\epsilon_f}}\right) e^{j\epsilon_{f_e}} \tag{30}$$

where

$$\epsilon_{fe} = \epsilon_f + \tan^{-1} \frac{R \sin \epsilon_f}{1 - R \cos \epsilon_f} \tag{31a}$$

$$= \epsilon_f + \sin^{-1} \frac{R \sin \epsilon_f}{\sqrt{(1-R \cos \epsilon_f)^2 + (R \sin \epsilon_f)^2}}$$

$$= \epsilon_f + \sin^{-1} \left(\left| \frac{v_f}{v_e} \right| \sin \epsilon_f \right) \tag{31b}$$

Information on the system characteristics can be obtained readily from a logarithmic graph of the open-loop response amplitude versus frequency. A slope of −1 on this type of plot represents a first-order term, since each doubling of frequency results in a halving of the magnitude. Similarly, a slope of −2 represents a second-order term since doubling the frequency reduces the magnitude to one-fourth its original magnitude, and so on. Inasmuch as the log of the amplitude is usually plotted on a uniform scale with frequency on a log scale, a unit of logarithmic amplitude is desirable. The decibel is commonly used because of the carry-over of feedback amplifier theory from communications engineering. The value in decibels in this case is equal to 20 times the logarithm of the amplitude ratio. However, there appears to be no valid reason for continuing its use, and a relatively new term, "loru," implying one logarithmic unit, is preferred. The value in lorus is simply the $\log_{10}$ of the amplitude ratio. Therefore, slopes of −1 and −2 correspond, respectively, to one or two lorus per frequency decade. In communications work and most servomechanism texts, these slopes would be referred to as −6 and −12 decibels per octave (reference 1, p. 241).

The open-loop response magnitude, then, may be expressed logarithmically in lorus as

$$\left| \frac{v_f}{v_e} \right|_{\log_{10}} = \log_{10} \left(\frac{R}{\sqrt{1+R^2-2R \cos \epsilon_f}} \right) \text{lorus} \tag{32}$$

For convenience, the expression in terms of decibels is also given as

$$\left| \frac{v_f}{v_e} \right|_{db} = 20 \log_{10} \left(\frac{R}{\sqrt{1+R^2-2R \cos \epsilon_f}} \right) \text{decibels} \tag{33}$$

The open-loop response for the servo system considered in the examples of the preceding sections was calculated from equations (31) and (32) using the values from the zero-rate signal curves. These calculated points are shown in figure 4. The straight-line asymptotes are

drawn with slopes of −1 and −2 lorus per decade representing the effects of first- and second-order terms, respectively, of the characteristic equation for the servo system. It is apparent that the servo behaves as a second-order system in the frequency range shown, except at very low frequencies. The falling off in amplitude may be attributed to the fact that the experimental phase-angle curve in figure 2 levels off at about 6° at low frequencies instead of approaching zero. It can be seen from equation (30) that too large a phase angle ϵ_f would cause a falling off of the open-loop amplitude v_f/v_e.

Autopilot-aircraft, open- and closed-loop responses from component values.- In synthesizing the autopilot-aircraft combination, the open-loop response is obtained by multiplying together the individual component transfer functions. In this case the servo-system closed-loop transfer function, in terms of δ/v_i, represents one of the components. The closed-loop response for the combination, again referring to figure 1, may be considered to be θ/θ_I where θ_I represents a hypothetical angular input to the system. In practice, a voltage v_I is used for the input and is made equivalent to θ_I by use of the displacement gyro constant k_g. The input voltage to the servo system is

$$\begin{aligned} v_i &= v_I - (v_g + v_r) \\ &= v_I - (k_g\theta + P_rA_r\theta) \end{aligned}$$

The expression for the forward part of the loop is

$$\theta = (\delta/v_i)\ (\theta/\delta)\ v_i$$

Combining these two equations to eliminate v_i

$$\theta = v_I\ (\delta/v_i)\ (\theta/\delta) - \theta\ (k_g + P_rA_r)\ (\delta/v_i)\ (\theta/\delta)$$

The closed-loop response is then

$$\frac{\theta}{v_I} = \frac{(\delta/v_i)\ (\theta/\delta)}{1 + (k_g + P_rA_r)\ (\delta/v_i)\ (\theta/\delta)}$$

In order to obtain the nondimensional closed-loop response θ/θ_I, v_I is made equal to $k_g\theta_I$ so that the preceding equation becomes

$$\frac{\theta}{\theta_I} = \frac{k_g\ (\delta/v_i)\ (\theta/\delta)}{1 + (k_g+P_rA_r)\ (\delta/v_i)\ (\theta/\delta)} \tag{34}$$

Although this equation can be used in its present form, it is more convenient to express it in terms including an over-all gain factor and

a nondimensional servo response. The concept of static control gearing k_p is introduced to represent the autopilot gain factor which is varied in the open-loop response for the purpose of altering the relative stability of the autopilot-aircraft combination. The static control gearing is defined as the ratio between control-surface deflection and angular attitude input to the autopilot at zero frequency. Thus

$$k_p = \left|\frac{\delta}{\theta}\right|_{f = 0}$$

From figure 1 it can be seen that $v_f = P_f k_f \delta$ and $v_g = k_g \theta$. Substituting for δ and θ

$$k_p = \left|\left(\frac{v_f}{P_f k_f}\right)\left(\frac{k_g}{v_g}\right)\right|_{f = 0}$$

Since, at zero frequency, $v_r = 0$ and $v_f = v_g$,

$$k_p = \frac{k_g}{P_f k_f} \tag{35}$$

As indicated in previous sections, the closed-loop servo-system response that is measured is v_f/v_i which is equal to $P_f k_f (\delta/v_i)$. By substitution of these relations in equation (34),

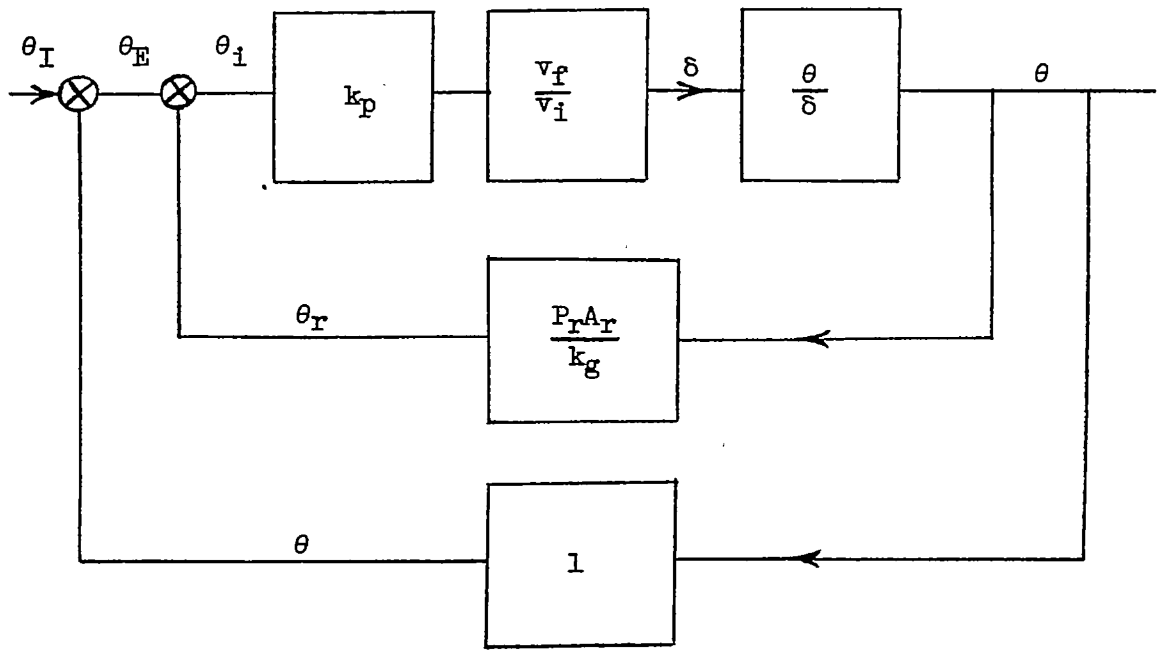
$$\frac{\theta}{\theta_I} = \frac{\frac{k_g}{P_f k_f}\ (v_f/v_i)\ (\theta/\delta)}{1 + \frac{k_g}{P_f k_f}\left[1 + \left(\frac{P_r A_r}{k_g}\right)\right](v_f/v_i)\ (\theta/\delta)}$$

or

$$\frac{\theta}{\theta_I} = \frac{k_p\ (v_f/v_i)\ (\theta/\delta)}{1 + k_p\left[1 + (P_r A_r/k_g)\right]\ (v_f/v_i)\ (\theta/\delta)} \tag{36}$$

The expression $\left[1 + (P_r A_r/k_g)\right](v_f/v_i)$ has been shown in equation (15) to be the autopilot rate response so that the experimental values can be used for this factor if the rate attenuator is at the value desired.

In terms of angular input, error, and output, equation (36) represents a system which may be diagrammed as follows:



This type of sketch is helpful in visualizing the autopilot-aircraft combination and its feedback loops in terms of angles with the control gearing k_p as the gain parameter. It should be noted, however, that varying k_p will also vary either the servo response v_f/v_i or the relative amount of rate to displacement feedback response P_rA_r/k_g, depending on whether P_fk_f or k_g is changed.

With a multiloop system such as shown in the sketch, it is possible to write more than one open-loop expression depending on where the loop is opened. However, in order to apply the Nyquist criterion to a polar plot of the open-loop response, it is necessary that any inner loops included in the function be stable. Hence the response θ/θ_E, which does contain an inner loop, cannot be used unless it is known that the inner loop θ_r/θ_i is stable. By breaking the loop at θ_i this difficulty may be eliminated. A single loop with two parallel arms then results and the desired open-loop response for the combination is simply the product of the various transfer functions around the loop, the transfer function of the two parallel arms being the sum of the two individual transfer functions. From the above figure it can be seen that the open-loop response after rearranging the order of the terms is

$$A_L = \frac{\theta+\theta_r}{\theta_i} = k_p \left(1 + \frac{P_rA_r}{k_g}\right)\left(\frac{v_f}{v_i}\right)\left(\frac{\theta}{\delta}\right) \tag{37}$$

Written in this fashion, the gain function k_p appears first followed by the autopilot transfer function $(1 + P_r A_r/k_g)(v_f/v_i)$ and the aircraft transfer function (θ/δ). By reference to figure 1, it will be recalled that the servo-system transfer function v_f/v_i contains an inner feedback loop. However, most autopilot servo systems are designed to be stable, and this analysis is then applicable.

Autopilot-aircraft, open-loop response from closed-loop response in flight.- When the response of a complete autopilot-aircraft combination is measured in flight, the closed-loop response in terms of θ/θ_I is normally the measured quantity. The open-loop response A_L may then be calculated provided the rate component in use, if any, is known. Equations (28) and (29) relate the open- and closed-loop responses for the servo loop. The same relations apply for the autopilot-aircraft loop diagrammed on page 18 if the corresponding quantities A_L and $(\theta+\theta_r)/\theta_I$ are used, respectively, in place of v_f/v_e and v_f/v_i. That is,

$$A_L = \frac{\dfrac{\theta+\theta_r}{\theta_I}}{1 - \left(\dfrac{\theta+\theta_r}{\theta_I}\right)} \tag{38}$$

Ordinarily θ/θ_I is measured rather than $(\theta+\theta_r)/\theta_I$. From the diagram on page 18 it is immediately apparent that θ_r is equal to $(P_r A_r/k_g)\theta$. By substitution of this value for θ_r in equation (38), the desired open-loop response is given by

$$A_L = \frac{\dfrac{\theta+(P_r A_r/k_g)\theta}{\theta_I}}{1 - \dfrac{\theta+(P_r A_r/k_g)\theta}{\theta_I}}$$

$$= \frac{\dfrac{\theta}{\theta_I}\left[1 + (P_r A_r/k_g)\right]}{1 - \dfrac{\theta}{\theta_I}\left[1 + (P_r A_r/k_g)\right]} \tag{39}$$

Autopilot-aircraft, error voltage.- In order to determine if a given autopilot and aircraft combination will operate within the linear range

for a given input voltage, it is again necessary to calculate the error voltage to the servo-amplifier attenuator in a manner similar to the case of the servo alone. Referring to figure 1 and adding the subscript c to designate the value for the combination, the equation for the error voltage is

$$\begin{aligned} v_{ec} &= v_i - v_f \\ &= v_i - \left(\frac{v_f}{v_i}\right) v_i \\ &= \left[v_I - (v_g + v_r)\right]\left[1 - (v_f/v_i)\right] \end{aligned} \tag{40}$$

The term $v_g + v_r$ may be considered the output of the over-all loop which includes the aircraft and which has unity feedback, the gyros now being in the forward part of the loop rather than the feedback portion. This output is then equal to the input v_I multiplied by the new closed-loop transfer function $(v_g + v_r)/v_I$. Since the open-loop response is unchanged regardless of where the loop is opened, the expression from equation (37) may be used and the new closed-loop response is

$$\frac{v_g + v_r}{v_I} = \frac{A_L}{1 + A_L}$$

Substituting for $v_g + v_r$ into equation (40)

$$\begin{aligned} v_{ec} &= \left(v_I - v_I \frac{A_L}{1 + A_L}\right)\left[1 - (v_f/v_i)\right] \\ &= \frac{v_I}{1 + A_L}\left[1 - (v_f/v_i)\right] \end{aligned} \tag{41}$$

Substituting $Re^{j\epsilon_f}$ for v_f/v_i and $|A_L|\, e^{j\epsilon_L}$ for A_L and resolving the numerator and denominator into their real and imaginary components

$$v_{ec} = v_I \frac{1 - R \cos \epsilon_f - j R \sin \epsilon_f}{1 + |A_L| \cos \epsilon_L + j\, |A_L| \sin \epsilon_L}$$

Since only the magnitude of the error voltage is of interest here, it is given by

$$\left|v_{ec}\right| = v_I \sqrt{\frac{1 + R^2 - 2R \cos \epsilon_f}{1 + \left|A_L\right|^2 + 2 \left|A_L\right| \cos \epsilon_L}} \tag{42}$$

By reference to equation (5), it can be seen that the expression for v_{ec} represents the error voltage of the servo system alone divided by a factor related to the autopilot-aircraft open-loop response. By inspection it can also be seen that at zero frequency the error voltage is zero and at very high frequencies, where the amplitude ratios are negligible, it is essentially equal to the input voltage. In between, however, it is possible for the error voltage to exceed considerably the input voltage.

Ames Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Moffett Field, Calif., March 15, 1951.

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2. Jones, Arthur L., and Briggs, Benjamin R.: A Survey of Stability Analysis Techniques for Automatically Controlled Aircraft. NACA TN 2275, 1951.

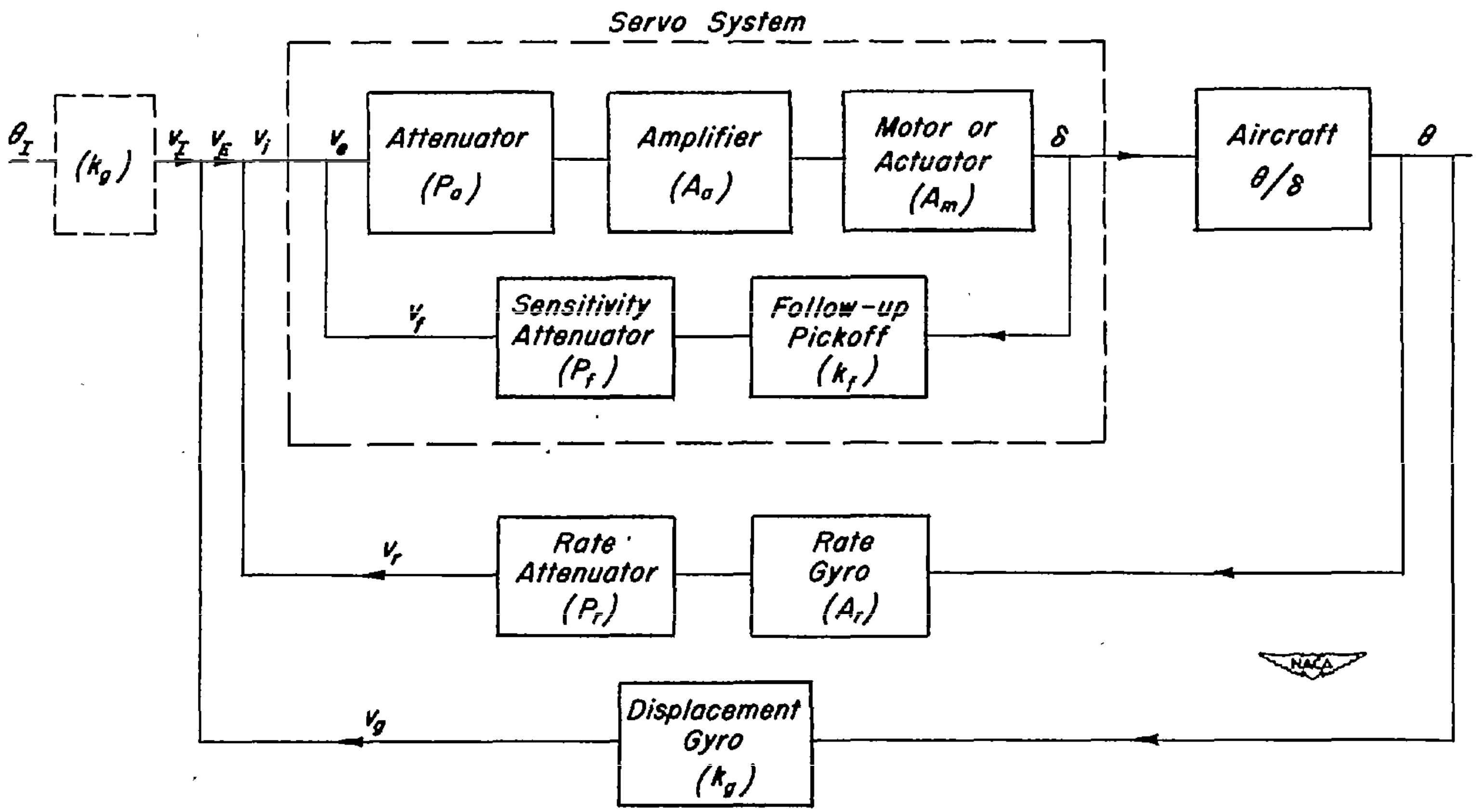

Figure 1.– Block diagram of one channel of a typical autopilot.

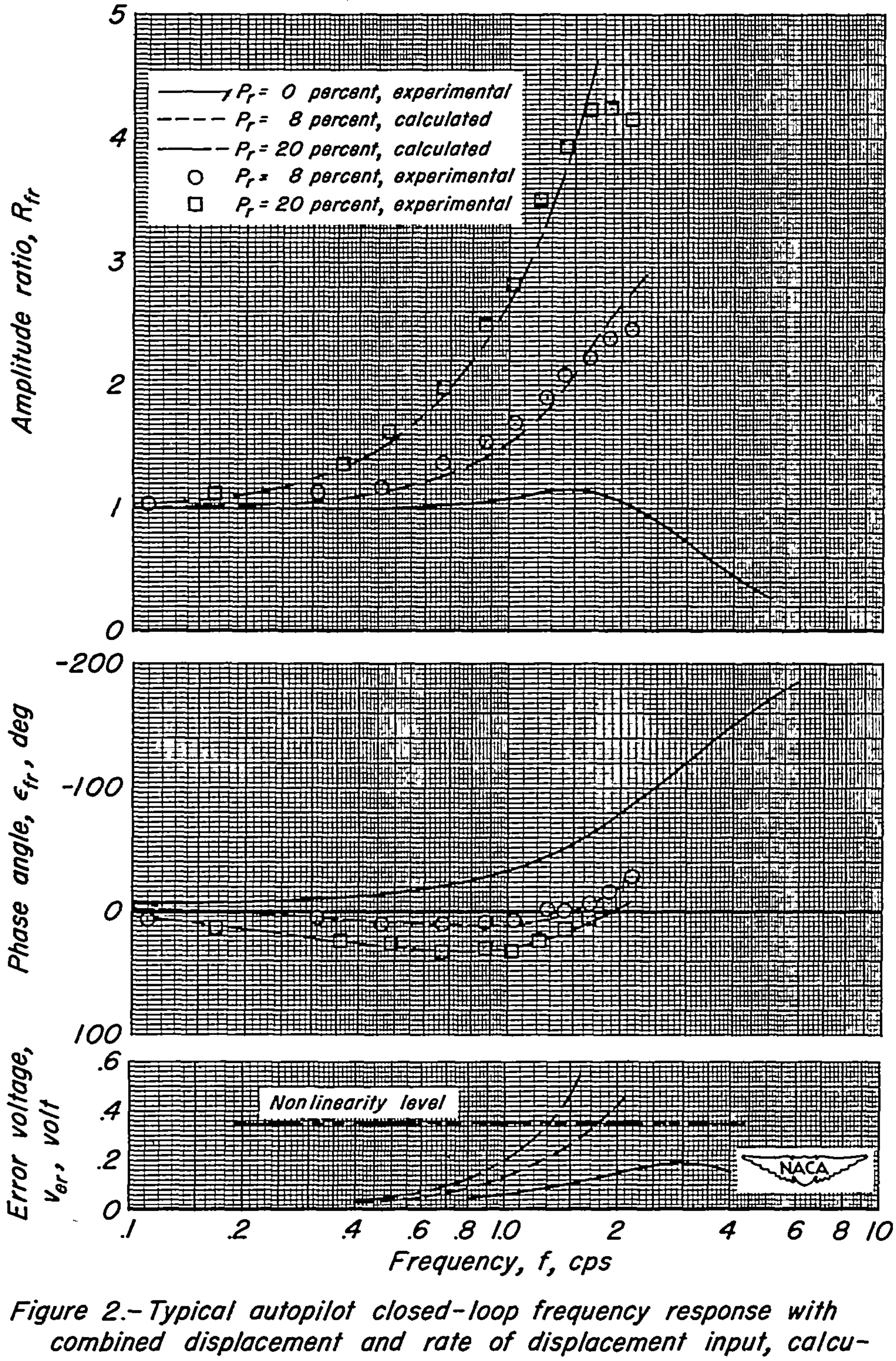

Figure 2.- Typical autopilot closed-loop frequency response with combined displacement and rate of displacement input, calculated and experimental. (v_g = ± 0.115 volt).

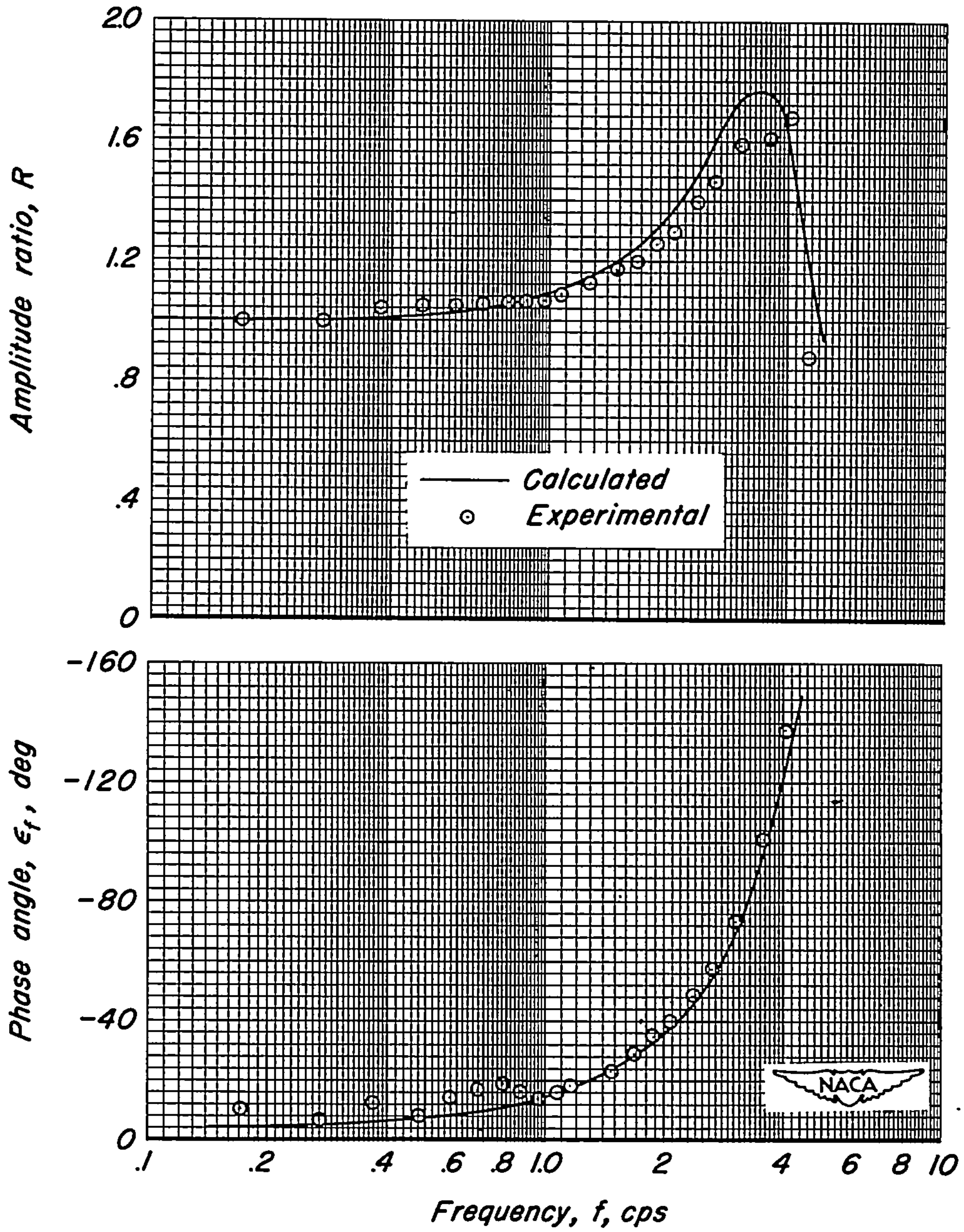

Figure 3.– Comparison of the calculated and experimental closed-loop frequency responses of the servo system for a change of gain.

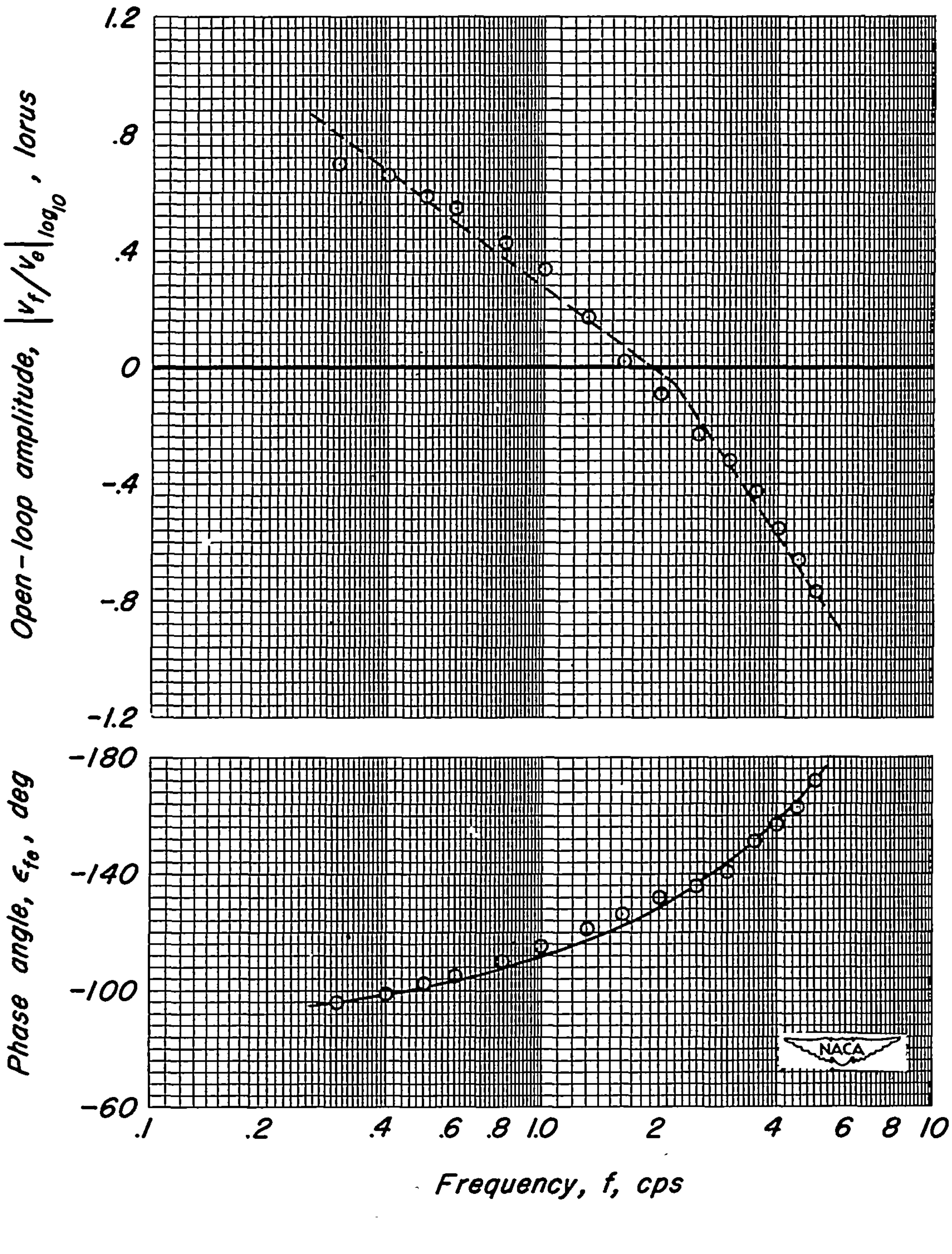

Figure 4.– Typical open-loop servo-system response calculated from closed-loop response.